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3.

a) If $\lim_{x\to c} g(x) = L$ then, for any $\varepsilon > 0$, $\exists\, \delta(\varepsilon)$ s.t.

$$0 < |x-c| < \delta(\varepsilon) \Rightarrow |g(x) - L| < \varepsilon .$$

If $L > 0$, ε can be taken with the value of L hence

$$0 < |x-c| < \delta(L) \Rightarrow |g(x) - L| < L .$$

When $g(x) = 0$, $|g(x) - L| < L \Rightarrow |-L| < L$ which is never true for $L > 0$.

When $g(x) < 0$, $g(x) - L < 0 \Rightarrow |g(x) - L| = -g(x) + L > L$ for $L > 0$ hence the definition fails for $g(x) < 0$.

Hence if $\lim_{x\to c} g(x) = L$ with $L > 0$, $g(x) > 0$ for some neighbourhood around c.

b) For a contradiction, assume $\exists\, y \in (c-\delta, c+\delta)$ s.t. $f(c) > f(y)$

If f is continuous at c then $\exists\, \delta$ s.t.

$$0 < |x-c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$$

For any $\varepsilon > 0$.

Take $\varepsilon = f(c) - f(y)$ then

$$x \in (c-\delta, c+\delta) \Rightarrow |f(x) - f(c)| < f(c) - f(y)$$
$$\Rightarrow f(y) - f(c) < f(x) - f(c) < f(c) - f(y)$$
$$\Rightarrow f(y) < f(x)$$

If $y \in (c-\delta, c)$ then f is decreasing in (y, c)

Then choose an x s.t. $y < x < c$ then $x \in (c-\delta, c+\delta)$ but $f(x) \leq f(y)$ as f is decreasing in (y, c).

Hence $y \notin (c-\delta, c)$.

If $y \in (c, c+\delta)$ then f is increasing in (c, y)

Then choose an x s.t. $c < x < y$ then $x \in (c-\delta, c+\delta)$ but $f(x) \leq f(y)$ as f is increasing in (c, y).

Hence $y \notin (c, c+\delta)$

$y \neq c$ as $f(c) > f(y)$ by assumption.

Therefore $y \notin (c-\delta, c+\delta)$ which is a contradiction.

Hence $f(x) \geq f(c)$ $\forall x \in (c-\delta, c+\delta)$

c) $f''(c) = \lim_{x\to c}\left(\frac{f'(x) - f'(c)}{x-c}\right)$

$= \lim_{x\to c}\left(\frac{f'(x)}{x-c}\right)$

$f''(c) > 0 \Rightarrow \lim_{x\to c}\left(\frac{f'(x)}{x-c}\right) > 0 \Rightarrow \frac{f'(x)}{x-c} > 0$ for all x in some neighbourhood of c by part (a)

If $x > c$, then $f'(x) > 0$ in the neighbourhood of c then f is strictly increasing in the neighbourhood of c by the Increasing-Decreasing Theorem.

If $x < c$, then $f'(x) < 0$ in the neighbourhood of c then f is strictly decreasing in the neighbourhood of c by the Increasing-Decreasing Theorem.

As f is differentiable at c, it is also continuous at c.

Then by part (b), $f(x) \geq f(c)$ for all x in some neighbourhood of c therefore c is a local minimum.