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i) a) $\{\neg\neg(P_1\to\neg P_2), \neg\neg P_1\} \vdash \neg\neg P_1$ by assumption

$\{\neg\neg(P_1\to\neg P_2), \neg\neg P_1\} \vdash (\neg\neg P_1 \to P_1)$ example 3.3.10

$\{\neg\neg(P_1\to\neg P_2), \neg\neg P_1\} \vdash P_1$ MP

$\{\neg\neg(P_1\to\neg P_2), \neg\neg P_1\} \vdash \neg\neg(P_1\to\neg P_2)$ by assumption

$\{\neg\neg(P_1\to\neg P_2), \neg\neg P_1\} \vdash (\neg\neg(P_1\to\neg P_2)\to(P_1\to\neg P_2))$ example 3.3.10

$\{\neg\neg(P_1\to\neg P_2), \neg\neg P_1\} \vdash (P_1\to\neg P_2)$ MP

$\{\neg\neg(P_1\to\neg P_2), \neg\neg P_1\} \vdash \neg P_2$ MP

$\{\neg\neg(P_1\to\neg P_2)\} \vdash (\neg\neg P_1\to\neg P_2)$ DT

$\vdash(\neg\neg(P_1\to\neg P_2)\to(\neg\neg P_1\to\neg P_2))$ DT

b) First showing $\vdash(\neg\alpha\to(\alpha\to\beta))$

$\{\alpha, \neg\alpha\} \vdash (\neg\alpha\to(\neg\beta\to\neg\alpha))$ L1

$\{\alpha, \neg\alpha\} \vdash \neg\alpha$ by assumption

$\{\alpha, \neg\alpha\} \vdash (\neg\beta\to\neg\alpha)$ MP

$\{\alpha, \neg\alpha\} \vdash ((\neg\beta\to\neg\alpha)\to(\alpha\to\beta))$ L3

$\{\alpha, \neg\alpha\} \vdash (\alpha\to\beta)$ MP

$\{\alpha, \neg\alpha\} \vdash \alpha$ by assumption

$\{\alpha, \neg\alpha\} \vdash \beta$ MP

$\{\neg\alpha\} \vdash (\alpha\to\beta)$ DT

$\vdash(\neg\alpha\to(\alpha\to\beta))$ DT

Then

$\{((\alpha\to\beta)\to\alpha), \neg\alpha\} \vdash ((\alpha\to\beta)\to\alpha)$ by assumption

$\{((\alpha\to\beta)\to\alpha), \neg\alpha\} \vdash \neg\alpha$ by assumption

$\{((\alpha\to\beta)\to\alpha), \neg\alpha\} \vdash (\neg\alpha\to(\alpha\to\beta))$ from above

$\{((\alpha\to\beta)\to\alpha), \neg\alpha\} \vdash (\alpha\to\beta)$ MP

$\{((\alpha\to\beta)\to\alpha), \neg\alpha\} \vdash \alpha$ MP

$\{((\alpha\to\beta)\to\alpha)\} \vdash (\neg\alpha\to\alpha)$ DT

$\{((\alpha\to\beta)\to\alpha)\} \vdash ((\neg\alpha\to\alpha)\to\alpha)$ Example 3.3.1

$\{((\alpha\to\beta)\to\alpha)\} \vdash \alpha$ MP

$\vdash(((\alpha\to\beta)\to\alpha)\to\alpha)$ DT

ii) For each i, t_i can either be

① $t_i = c_j$

② $t_i = x_j$

③ $t_i = f_j^{(k)}(t_1,\dots,t_k)$

If ①,

$v[t_i] = v[c_j] = \hat{c}_j$

Since w is also a valuation in M,

$w[t_i] = w[c_j] = \hat{c}_j = v[t_i]$

If ②,

$v[t_i] = v[x_j] = w[x_j]$ by assumption

If ③,

$v[t_i] = v[f_j^{(k)}(t_1,\dots,t_k)] = \hat{f}_j^{(k)}(v[t_1],\dots,v[t_k])$

Since w is also a valuation in M,

$w[t_i] = w[f_j^{(k)}(t_1,\dots,t_k)] = \hat{f}_j^{(k)}(w[t_1],\dots,w[t_k])$

If a term t_n in $\hat{f}_j^{(k)}$ is of the form ① or ② then from the above, $v[t_i] = w[t_i]$

If the term is of the form ③ then the valuation can be done recursively until only terms of the form ① or ② exist in the function.

Then

$v[t_i] = \hat{f}_j^{(k)}(v[t_1],\dots,v[t_k]) = \hat{f}_j^{(k)}(w[t_1],\dots,w[t_k]) = w[t_i]$

Hence

$v[t_i] = w[t_i]\ \forall i$

ii cont.)

Base case:

If α is an atomic formula then

① $\alpha = R_i(t_1, t_2, \dots, t_k)$ or

② $\alpha = t_1 \approx t_2$

If ①,

$v[\alpha] = T \Leftrightarrow (v[t_1],\dots,v[t_k]) \in \hat{R}_i$.

Since w is also a valuation in M,

$w[\alpha] = T \Leftrightarrow (w[t_1],\dots,w[t_k]) \in \hat{R}_i$

And $v[t_i] = w[t_i]\ \forall i$ hence

$(w[t_1],\dots,w[t_k]) \in \hat{R}_i \Leftrightarrow (v[t_1],\dots,v[t_k]) \in \hat{R}_i$

so $w[\alpha] = T \Leftrightarrow v[\alpha] = T$.

If ②,

$v[\alpha] = T \Leftrightarrow v[t_1] = v[t_2]$.

$v[t_i] = w[t_i]\ \forall i$ hence

$v[t_1] = v[t_2] \Leftrightarrow w[t_1] = w[t_2]$.

Therefore

$v[\alpha] = T \Leftrightarrow w[\alpha] = T$.

Inductive hypothesis:

Assuming for $\beta, \gamma \in \mathcal{L}^*$

$v[\beta] = T \Leftrightarrow w[\beta] = T$ and $v[\gamma] = T \Leftrightarrow w[\gamma] = T$.

Inductive step.

Let α be a formula of one of the forms

① $\alpha = \neg\beta$

② $\alpha = (\beta\to\gamma)$

③ $\alpha = \forall x_i\,\beta$

If α is of form ①,

$v[\alpha] = v[\neg\beta]$ and $w[\alpha] = w[\neg\beta]$.

Then $v[\alpha] = T \Leftrightarrow v[\beta] = F$

and $w[\alpha] = T \Leftrightarrow w[\beta] = F$.

From the inductive hypothesis,

$v[\beta] = F \Leftrightarrow w[\beta] = F$

so

$v[\alpha] = T \Leftrightarrow w[\alpha] = T$.

If α is of form ②,

$v[\alpha] = v[\beta\to\gamma]$ and $w[\alpha] = w[\beta\to\gamma]$.

Then $v[\alpha] = F \Leftrightarrow v[\beta] = T$ and $v[\gamma] = F$

and $w[\alpha] = F \Leftrightarrow w[\beta] = T$ and $w[\gamma] = F$.

From the inductive hypothesis,

$v[\beta] = T \Leftrightarrow w[\beta] = T$ and $v[\gamma] = F \Leftrightarrow w[\gamma] = F$.

Hence

$v[\alpha] = F \Leftrightarrow w[\alpha] = F$

so $v[\alpha] = T \Leftrightarrow w[\alpha] = T$.

If α is of form ③,

$v[\alpha] = v[\forall x_i\,\beta]$ and $w[\alpha] = w[\forall x_i\,\beta]$.

Then $v[\alpha] = T \Leftrightarrow r[\beta] = T$ and $w[\alpha] = T \Leftrightarrow p[\beta] = T$

for all valuations such that $v[x_j] = r[x_j]$ and $w[x_j] = p[x_j]$.

But $v[x_j] = w[x_j]$ by assumption hence in these cases, $v[x_j] = w[x_j] = r[x_j] = p[x_j]$.

Hence

$v[\alpha] = T \Leftrightarrow w[\alpha] = T$.

Since $v[\alpha] = T \Leftrightarrow w[\alpha] = T$ holds for any extension of a formula where it already holds and it holds for all atomic formulae,

$v[\alpha] = T \Leftrightarrow w[\alpha] = T\ \ \forall \alpha \in \mathcal{L}^*$.

Therefore $M \vDash \alpha[v] \Leftrightarrow M \vDash \alpha[w]$.