20857212

a) S_{3+} is the survival function for adult frogs on each time-step as no limit on the age of the frogs is used in the model.

$S_1 = 0.018$

$S_2 = 0.015$

$S_3 = 0.8$

$S_{3+} = 0.9$

$M_1 = 0$

$M_2 = 0$

$M_3 = 2000$

b) The eigenvalues are given by

$$\begin{vmatrix} -\lambda & 0 & 36 \\ 0.015 & -\lambda & 0 \\ 0 & 0.8 & 0.9-\lambda \end{vmatrix} = 0 \Rightarrow -\lambda(-0.9\lambda + \lambda^2) + 36(0.012) = 0$$

$$\Rightarrow -\lambda^3 + 0.9\lambda^2 + 0.432 = 0$$

$$\Rightarrow \lambda^3 - 0.9\lambda^2 - 0.432 = 0 \quad ①$$

Substituting $\lambda = 1.2$ into left-hand side of ①,

$$(1.2)^3 - 0.9(1.2)^2 - 0.432 = 1.728 - 1.296 - 0.432 = 0$$

Hence $\lambda_1 = 1.2$ is an eigenvalue.

$$\begin{array}{r} \lambda^2 + 0.3\lambda + 0.36 \\ \lambda - 1.2 \,\big)\, \lambda^3 - 0.9\lambda^2 + 0\lambda - 0.432 \\ \lambda^3 - 1.2\lambda^2 \\ 0.3\lambda^2 + 0\lambda \\ 0.3\lambda^2 - 0.36\lambda \\ 0.36\lambda - 0.432 \\ 0.36\lambda - 0.432 \\ 0 \end{array}$$

So ① $\Rightarrow (\lambda - 1.2)(\lambda^2 + 0.3\lambda + 0.36) = 0$

$$\text{So } \lambda_{2,3} = \frac{-0.3 \pm \sqrt{0.09 - 1.44}}{2} = -0.15 \pm i\sqrt{0.3375}$$

Then $\lambda_2 = -0.15 + i\sqrt{0.3375}$ and $\lambda_3 = -0.15 - i\sqrt{0.3375}$.

$$|\lambda_2| = |\lambda_3| = \sqrt{0.15^2 + 0.3375} = 0.6 < |\lambda_1| = 1.2$$

Hence λ_1 is the largest magnitude eigenvalue.

c) The corresponding eigenvetor is given by

$$\begin{pmatrix} -1.2 & 0 & 36 \\ 0.015 & -1.2 & 0 \\ 0 & 0.8 & -0.3 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \underline{0}$$

This gives the system of equations

$$\begin{aligned} -1.2z_1 \qquad\quad + 36z_3 &= 0 \\ 0.015z_1 - 1.2z_2 \qquad\quad &= 0 \\ 0.8z_2 - 0.3z_3 &= 0 \end{aligned}$$

So rearranging gives

$z_2 = 0.375z_3$

$z_1 = 80z_2 = 80(0.375z_3) = 30z_3$

So the unnormalised eigenvector is

$$\underline{v}_1 = \begin{pmatrix} 30 \\ 0.375 \\ 1 \end{pmatrix}$$

This has magnitude

$$d = \sqrt{\underline{v}_1 \cdot \underline{v}_1} = \sqrt{30^2 + 0.375^2 + 1} = \sqrt{901.140625}$$

So the nomalised eigenvector is

$$\hat{\underline{v}}_1 = \frac{1}{\sqrt{901.140625}} \begin{pmatrix} 30 \\ 0.375 \\ 1 \end{pmatrix} \approx \begin{pmatrix} 0.9994 \\ 0.0125 \\ 0.0333 \end{pmatrix}$$

d)

t	T	J	A
0	0	0	5
1	180	0	4.5
2	162	2.7	4.05
3	145.8	2.43	5.805
4	208.98	2.187	7.1685
5	258.066	3.1347	8.20125
6	295.245	3.87099	9.888885
7	355.99486	4.428675	11.9967885
8	431.884386	5.3399979	14.34004965
9	516.2417874	6.47826579	17.17804301
10	618.4095482	7.743626811	20.64285134

So the unnormalised vector is approximately

$$\underline{z}_1 = \begin{pmatrix} 618.4095 \\ 7.7436 \\ 20.6429 \end{pmatrix}$$

Then the magnitude is

$$h = \sqrt{\underline{z}_1 \cdot \underline{z}_1} = \sqrt{618.4095^2 + 7.7436^2 + 20.6429^2} = \sqrt{382916.4024}$$

So the normalised vector is

$$\hat{\underline{z}}_1 = \frac{1}{\sqrt{382916.4024}} \begin{pmatrix} 618.4095 \\ 7.7436 \\ 20.6429 \end{pmatrix} \approx \begin{pmatrix} 0.9994 \\ 0.0125 \\ 0.0333 \end{pmatrix}$$

Hence the long-term age structure and the theoretical structure agree.