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a) $x_0=0$ is an ordinary point of the ODE as 0 and $-x$ are both analytic for any x_0.

Therefore we seek a solution to the ODE in the form,

$$y(x)=\sum_{r=0}^{\infty} a_r x^r.$$

So $y'(x)=\sum_{r=1}^{\infty} r a_r x^{r-1}$

and $y''(x)=\sum_{r=2}^{\infty} r(r-1)a_r x^{r-2}$.

Substituting these into the ODE gives,

$$\sum_{r=2}^{\infty} r(r-1)a_r x^{r-2} - x\sum_{r=0}^{\infty} a_r x^r = 0$$

$$\Leftrightarrow \sum_{r=2}^{\infty} r(r-1)a_r x^{r-2} - \sum_{r=0}^{\infty} a_r x^{r+1} = 0$$

In the first series, let $m=r-1$, in the other series, let $m=r+1$ then

$$\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m - \sum_{m=1}^{\infty} a_{m-1}x^m = 0$$

As $\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m = 2a_2 + \sum_{m=1}^{\infty}(m+2)(m+1)a_{m+2}x^m$, then

$$2a_2 + \sum_{m=1}^{\infty}(m+2)(m+1)a_{m+2}x^m - \sum_{m=1}^{\infty} a_{m-1}x^m = 0$$

$$\Leftrightarrow 2a_2 + \sum_{m=1}^{\infty}\left[(m+2)(m+1)a_{m+2} - a_{m-1}\right]x^m = 0$$

Then by Section 3.1.4,

$x^0: 2a_2 = 0 \Rightarrow a_2 = 0$

$x^m: (m+2)(m+1)a_{m+2} - a_{m-1} = 0 \Rightarrow a_{m+2} = \frac{a_{m-1}}{(m+2)(m+1)}$

This is a recurrence relation:

$m=1: a_3 = \frac{a_0}{3(2)} = \frac{a_0}{6}$

$m=2: a_4 = \frac{a_1}{4(3)} = \frac{a_1}{12}$

$m=3: a_5 = \frac{a_2}{5(4)} = \frac{a_2}{20} = 0$

$m=4: a_6 = \frac{a_3}{6(5)} = \frac{a_3}{30} = \frac{a_0}{180}$

$m=5: a_7 = \frac{a_4}{7(6)} = \frac{a_4}{42} = \frac{a_1}{504}$

etc.

This gives the general solution,

$$y(x) = a_0\left(1+\tfrac{1}{6}x^3+\tfrac{1}{180}x^6+\dots\right) + a_1\left(x+\tfrac{1}{12}x^4+\tfrac{1}{504}x^7+\dots\right)$$

b) $y(0)=0 \Rightarrow a_0=0$

$y'(x) = a_0\left(\tfrac{1}{2}x^2+\tfrac{1}{30}x^5+\dots\right) + a_1\left(1+\tfrac{1}{3}x^3+\tfrac{1}{72}x^5+\dots\right)$

Then $y'(0)=1 \Rightarrow a_1=1$

Hence the solution to the initial value problem is

$y(x) = 1+\frac{1}{12}x^4+\frac{1}{504}x^7+\dots$

$\quad = 1+\frac{2}{24}x^4+\frac{10}{5040}x^7+\dots$

$\quad = 1+\frac{2}{4!}x^4+\frac{5\times2}{7!}x^7+\dots$

e) All three solutions are similarly accurate about $x=0$ however the solution found in part (b) diverges from the other solutions quickly away from $x=0$. This is because the solution only goes up to the x^7 term hence it will change drastically from the solution from part (c), with terms up to x^{40}, as x gets larger. The numerical solutions from Maple most likely uses an even more precise equation/algorithm than the part (c) solution therefore it eventually diverges from the numerical solutions around $x=-6$. When $x>0$ the ODE solution seems to start resembling an exponential while for $x<0$ it resembles an oscillating function such as $\cos(x)$ or $\sin(x)$.