20857212

a) $a = 4t^2$

$b = 0$

$c = -1$

Then $b^2 - 4ac = 0^2 - 4(4t^2)(-1)$

$= 16t^2$.

$16t^2 > 0 \ \forall t \in [1, \infty)$ hence the PDE is hyperbolic throughout its domain.

b) Note that
$\frac{\partial \xi}{\partial x} = 1$, $\frac{\partial \xi}{\partial t} = -2t$, $\frac{\partial \eta}{\partial x} = 1$, $\frac{\partial \eta}{\partial t} = 2t$.

Then by the chain rule,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}$$

$$\text{so } \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial x}\right)\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial x}\right)\frac{\partial \eta}{\partial x}$$

$$= \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right) + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right)$$

$$= \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}$$

Also $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} = 2t\frac{\partial u}{\partial \eta} - 2t\frac{\partial u}{\partial \xi}$

$$\text{so } \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)$$

$$= \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial t}\right)\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial t}\right)\frac{\partial \eta}{\partial t}$$

$$= -2t\frac{\partial}{\partial \xi}\left(2t\frac{\partial u}{\partial \eta} - 2t\frac{\partial u}{\partial \xi}\right) + 2t\frac{\partial}{\partial \eta}\left(2t\frac{\partial u}{\partial \eta} - 2t\frac{\partial u}{\partial \xi}\right)$$

$$= 4t^2\frac{\partial^2 u}{\partial \xi^2} - 8t^2\frac{\partial^2 u}{\partial \xi \partial \eta} + 4t^2\frac{\partial^2 u}{\partial \eta^2}$$

Substituting these into the PDE,

$$4t^2\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} + \frac{1}{t}\frac{\partial u}{\partial t} = 0 \Rightarrow 4t^2\left(\frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}\right) - \left(4t^2\frac{\partial^2 u}{\partial \xi^2} - 8t^2\frac{\partial^2 u}{\partial \xi \partial \eta} + 4t^2\frac{\partial^2 u}{\partial \eta^2}\right) + \frac{1}{t}\left(2t\frac{\partial u}{\partial \eta} - 2t\frac{\partial u}{\partial \xi}\right) = 0$$

$$\Rightarrow 16t^2\frac{\partial^2 u}{\partial \xi \partial \eta} + 2\frac{\partial u}{\partial \eta} - 2\frac{\partial u}{\partial \xi} = 0$$

$$\Rightarrow 8t^2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial u}{\partial \eta} - \frac{\partial u}{\partial \xi} = 0$$

$$\Rightarrow 4(\eta - \xi)\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial u}{\partial \eta} - \frac{\partial u}{\partial \xi} = 0$$

c) From the general solutions and initial conditions,

$u(x,1) = r(x) = f(x-1) + g(x+1)$ ①

From the general solution,

$\frac{\partial u}{\partial t} = -2tf'(x-t^2) + 2tg'(x+t^2)$

So by the initial conditions,

$\frac{\partial u}{\partial t}(x,1) = 0 = -2f'(x-1) + 2g'(x-1)$.

Integrating this from x_0 to x where x_0 is arbitrary,

$-2f(x-1) + 2g(x+1) = \int_{x_0}^{x} 0 \, d\phi$

$\Rightarrow -f(x-1) + g(x+1) = 0$ ②

Then ① and ② form a linear system,

$g(x) = \frac{1}{2}r(x+1)$

$f(x) = \frac{1}{2}r(x-1)$.

So substituting these into the general solution,

$u(x,t) = \frac{1}{2}r((x+1)-t^2) + \frac{1}{2}r((x-1)+t^2)$

$= \frac{1}{2}[r(x+1-t^2) + r(x-1+t^2)]$

d) $u(x,1) = \frac{1}{2}[r(x+1-1) + r(x-1+1)]$

$= r(x) = e^{-x^2}$

$u(x,2) = \frac{1}{2}[r(x+1-4) + r(x-1+4)]$

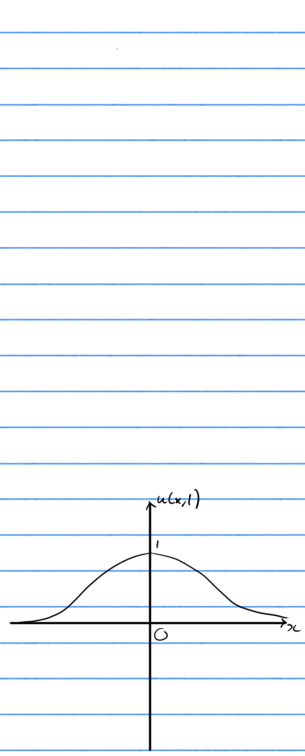
$= \frac{1}{2}[r(x-3) + r(x+3)]$

$= \frac{1}{2}\left(e^{-(x-3)^2} + e^{-(x+3)^2}\right)$

$u(x,3) = \frac{1}{2}\left(e^{-(x-8)^2} + e^{-(x+8)^2}\right)$

$u(x,4) = \frac{1}{2}\left(e^{-(x-15)^2} + e^{-(x+15)^2}\right)$

The result seems to be 2 bell curves that travel in opposite directions along the x-axis as t increases. The PDE then seems to describe functions which travel along the x-axis in opposite directions while being added together.



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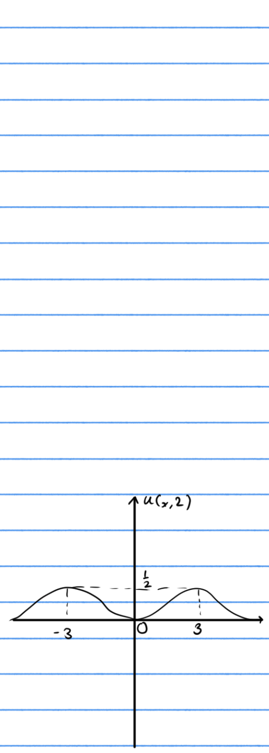

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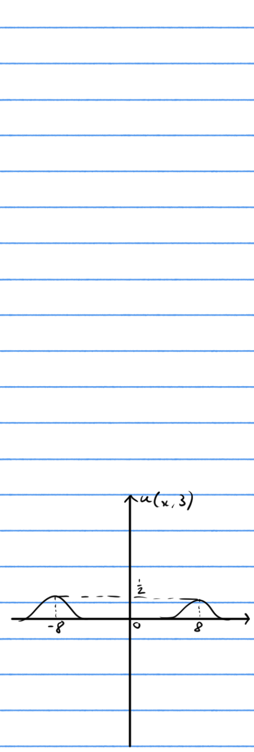

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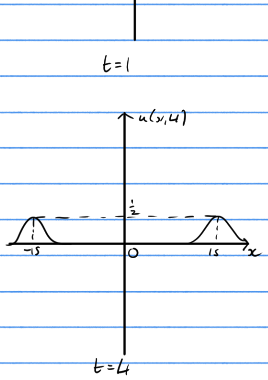

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