20857212

a) Define a Fourier transform in the t-direction,

$\mathcal{F}_t[u(x,t)] = \int_{-\infty}^{\infty} u(x,t)e^{-i\omega t}dt = U(x,\omega)$

Transforming the PDE in the t-direction,

$\mathcal{F}_t\left[\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial t} + \alpha u\right] = \mathcal{F}[0]$

$\Rightarrow \mathcal{F}_t\left[\frac{\partial u}{\partial x}\right] + 2\mathcal{F}_t\left[\frac{\partial u}{\partial t}\right] + \alpha\mathcal{F}[u] = 0$ by linearity

$\Rightarrow \frac{\partial}{\partial x}U(x,\omega) + 2i\omega U(x,\omega) + \alpha U(x,\omega) = 0$ by Theorems 16.1 and 16.2

$\Rightarrow \frac{\partial}{\partial x}U(x,\omega) + (2i\omega + \alpha)U(x,\omega) = 0$

The auxiliary equation is

$m + 2i\omega + \alpha = 0 \Rightarrow m = -2i\omega - \alpha$ so

$U(x,\omega) = A(\omega)e^{-(2i\omega+\alpha)x}$ ①

Transforming the boundary condition to find $A(\omega)$,

$\mathcal{F}_t[u(0,t)] = \mathcal{F}_t[f(t)] \Rightarrow U(0,\omega) = F(\omega)$

where $F(\omega) = \mathcal{F}[f(t)]$.

From ①,

$U(0,t) = A(\omega)e^{0} = A(\omega)$

Hence

$A(\omega) = F(\omega)$ so

$U(x,\omega) = F(\omega)e^{-(2i\omega+\alpha)x}$

Then define $G(x,\omega) = e^{-(2i\omega+\alpha)x}$ so

$U(x,\omega) = F(\omega)G(x,\omega)$

b) From the standard transforms,

$\mathcal{F}^{-1}[e^{-i\omega\tau}] = \delta(t-\tau)$

$G(x,\omega) = e^{-2i\omega x - \alpha x} = e^{-\alpha x}e^{-2i\omega x}$

$e^{-\alpha x}$ only depends on x hence with $\tau = 2x$ in the standard transform,

$g(x,t) = \mathcal{F}_t^{-1}[G(x,\omega)] \underset{\text{by linearity}}{=} e^{-\alpha x}\mathcal{F}_t^{-1}[e^{-2i\omega x}] = e^{-\alpha x}\delta(t-2x)$.

c) $\mathcal{F}_t^{-1}[U(x,\omega)] = \mathcal{F}_t^{-1}[F(\omega)G(x,\omega)]$

$\Rightarrow u(x,t) = (f * g)(x,t)$

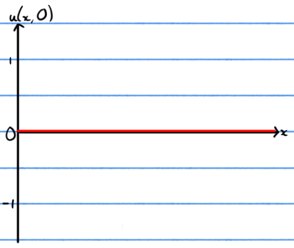
$= \int_{-\infty}^{\infty} f(\varsigma)g(x,t-\varsigma)d\varsigma$

$= e^{-\alpha x}\int_{-\infty}^{\infty} f(\varsigma)\delta(t-2x-\varsigma)d\varsigma$

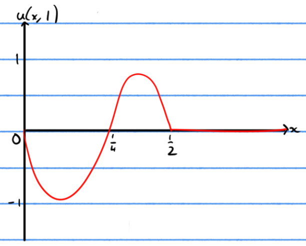
$= e^{-\alpha x}f(t-2x)$ by the sifting property of the delta function.

d) $u(x,t) = e^{-\alpha x}H(t-2x)\sin(2\pi(t-2x))$ $x \geq 0$

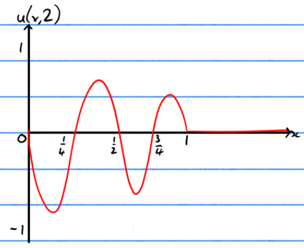
$= \begin{cases} e^{-\alpha x}\sin(2\pi(t-2x)) & x < \frac{1}{2}t \\ 0 & x \geq \frac{1}{2}t \end{cases}$



t=0



t=1



t=2

e) Up to $x = \frac{1}{2}t$, the solutions exhibit sinusoidal behaviour which reduces in amplitude as x increases. Past $x = \frac{1}{2}t$ the solution is 0. A physical system might be sound waves being produced and propagating through a medium where $x = 0$ is the point where the sound is produced.