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a) By definition,

$$\mathcal{L}[e^{-\frac{1}{4}t^2}] = \int_0^\infty e^{-\frac{1}{4}t^2} e^{-st}\,dt$$

$$= \int_0^\infty e^{-\frac{1}{4}t^2 - st}\,dt$$

$$= \int_0^\infty e^{-((\frac{t}{2})^2 + 2s(\frac{t}{2}))}\,dt$$

$$= \int_0^\infty e^{-((\frac{t}{2}+s)^2 - s^2)}\,dt \quad \text{completing the square}$$

$$= \int_0^\infty e^{s^2 - (\frac{t}{2}+s)^2}\,dt$$

$$= e^{s^2}\int_0^\infty e^{-(\frac{t}{2}+s)^2}\,dt \quad \text{as } e^{s^2} \text{ does not depend on } t$$

Let $y = \frac{t}{2}+s$ then $\frac{dy}{dt} = \frac{1}{2}$ so $dt = 2dy$ and

$t=0 \Rightarrow y=s$ and as $t\to\infty$, $y\to\infty$ so

$$\mathcal{L}[e^{-\frac{1}{4}t^2}] = e^{s^2}\int_s^\infty e^{-y^2}\,2dy$$

$$= 2e^{s^2}\int_s^\infty e^{-y^2}\,dy$$

$$= \sqrt{\pi}\, e^{s^2} \frac{2}{\sqrt{\pi}} \int_s^\infty e^{-y^2}\,dy$$

$$= \sqrt{\pi}\, e^{s^2}\operatorname{erfc}(s) \quad \text{from the definition of erfc.}$$

b) Let $f(t) = e^{-\frac{1}{4}t^2}$ so $\mathcal{L}[f(t)] = F(s) = \sqrt{\pi}\, e^{s^2}\operatorname{erfc}(s)$.

Then,

Note $e^{-t^2} = e^{-\frac{1}{4}(2t)^2} = f(2t)$ so

$$f_1(t) = f(2t) \Rightarrow \mathcal{L}[f_1(t)] = \mathcal{L}[f(2t)]$$

$$= \frac{1}{2}F(\frac{s}{2}) \quad \text{by the scaling property}$$

$$= \frac{\sqrt{\pi}}{2} e^{(\frac{s}{2})^2}\operatorname{erfc}(\frac{s}{2}) \quad s\in\mathbb{C}$$

Note $e^{st-\frac{1}{4}t^2} = e^{25-(\frac{t}{2}-5)^2} = e^{25}e^{-(\frac{t}{2}-5)^2} = e^{25}e^{-\frac{1}{4}(t-10)^2} = e^{25}f(t-10)$ so

$$f_2(t) = e^{25}f(t-10) \Rightarrow \mathcal{L}[f_2(t)] = \mathcal{L}[e^{25}f(t-10)]$$

$$\Rightarrow \mathcal{L}[f_2(t)] = e^{25}\mathcal{L}[f(t-10)] \quad \text{by linearity}$$

$$= e^{25}e^{-10s}\mathcal{L}[f(t)] \quad \text{by the second shifting property}$$

$$= e^{25}e^{-10s}F(s)$$

$$= e^{25}e^{-10s}\sqrt{\pi}\, e^{s^2}\operatorname{erfc}(s)$$

$$= \sqrt{\pi}\, e^{(s-5)^2}\operatorname{erfc}(s) \quad s\in\mathbb{C}$$

Note $-\frac{1}{2}t\, e^{-\frac{1}{4}t^2} = f'(t)$ so

$$f_3(t) = f'(t) \Rightarrow \mathcal{L}[f_3(t)] = \mathcal{L}[f'(t)]$$

$$= sF(s) - f(0) \quad \text{by the Laplace transform for ordinary derivatives}$$

$$= s\sqrt{\pi}\, e^{s^2}\operatorname{erfc}(s) - 1 \quad s\in\mathbb{C}$$

c) $\mathcal{L}[x(t)] = \int_0^\infty x(t)e^{-st}\,dt = X(s)$ then ①

by Theorem 20.2,

$$\mathcal{L}[\tfrac{d^4x}{dt^4}] = s^4X(s) - s^3x(0) - s^2x'(0) - sx''(0) - x'''(0)$$

$$= s^4X(s) - 1 \quad \text{from the initial conditions.} \quad ②$$

Applying the transform on the ODE,

$$\mathcal{L}[\tfrac{d^4x}{dt^4} - \Omega^4 x] = \mathcal{L}[0]$$

$$\Rightarrow \mathcal{L}[\tfrac{d^4x}{dt^4}] - \Omega^4\mathcal{L}[x] = 0 \quad \text{by linearity}$$

$$\Rightarrow s^4X(s) - 1 - \Omega^4X(s) = 0 \quad \text{from ① and ②}$$

$$\Rightarrow X(s)(s^4 - \Omega^4) = 1$$

$$\Rightarrow X(s) = \frac{1}{s^4-\Omega^4} = \frac{1}{(s^2-\Omega^2)(s^2+\Omega^2)}$$

Applying partial fractions,

$$X(s) = \frac{As+B}{s^2-\Omega^2} + \frac{Cs+D}{s^2+\Omega^2} \Rightarrow 1 = (As+B)(s^2+\Omega^2) + (Cs+D)(s^2-\Omega^2)$$

For this to be true for all s,

$$\left.\begin{aligned} A+C&=0\\ B+D&=0\\ \Omega^2A-\Omega^2C&=0\\ \Omega^2B-\Omega^2D&=1 \end{aligned}\right\} A=0,\ C=0,\ B=\frac{1}{2\Omega^2},\ D=-\frac{1}{2\Omega^2}$$

So

$$X(s) = \frac{1}{2\Omega^2}\left(\frac{1}{s^2-\Omega^2}\right) - \frac{1}{2\Omega^2}\left(\frac{1}{s^2+\Omega^2}\right)$$

Note $\frac{1}{s^2-\Omega^2} = \frac{1}{(s-\Omega)(s+\Omega)}$,

Apply partial fractions,

$$\frac{1}{(s-\Omega)(s+\Omega)} = \frac{A}{s-\Omega} + \frac{B}{s+\Omega} \Rightarrow 1 = A(s+\Omega) + B(s-\Omega)$$

For this to be true for all s,

$$\left.\begin{aligned} A+B&=0\\ \Omega A-\Omega B&=1 \end{aligned}\right\} A=\frac{1}{2\Omega},\ B=-\frac{1}{2\Omega}$$

So

$$\frac{1}{(s-\Omega)(s+\Omega)} = \frac{1}{2\Omega}\left(\frac{1}{s-\Omega} - \frac{1}{s+\Omega}\right)$$

c cont.)

Then,

$$X(s) = \frac{1}{4\Omega^3}\left(\frac{1}{s-\Omega} - \frac{1}{s+\Omega}\right) - \frac{1}{2\Omega^2}\left(\frac{1}{s^2+\Omega^2}\right)$$

So

$$x(t) = \mathcal{L}^{-1}[X(s)] = \mathcal{L}^{-1}\left[\frac{1}{4\Omega^3}\left(\frac{1}{s-\Omega} - \frac{1}{s+\Omega}\right) - \frac{1}{2\Omega^2}\left(\frac{1}{s^2+\Omega^2}\right)\right]$$

$$= \frac{1}{4\Omega^3}\left(\mathcal{L}^{-1}[\tfrac{1}{s-\Omega}] - \mathcal{L}^{-1}[\tfrac{1}{s+\Omega}]\right) - \frac{1}{2\Omega^2}\left(\mathcal{L}^{-1}[\tfrac{1}{s^2+\Omega^2}]\right) \quad \text{by linearity}$$

Note,

$\mathcal{L}^{-1}[\frac{1}{s-\Omega}] = e^{\Omega t}H(t)$ from the standard transforms and the first shifting property.

$\mathcal{L}^{-1}[\frac{1}{s+\Omega}] = e^{-\Omega t}H(t)$ from the standard transforms and the first shifting property.

$\mathcal{L}^{-1}[\frac{1}{s^2+\Omega^2}] = \mathcal{L}^{-1}[\frac{1}{\Omega^2}\frac{1}{(\frac{s}{\Omega})^2+1}] = \frac{1}{\Omega}\mathcal{L}^{-1}[\frac{1}{\Omega}\frac{1}{(\frac{s}{\Omega})^2+1}]$ by linearity

$= \frac{1}{\Omega}H(t)\sin(\Omega t)$ from the standard transforms and the scaling property.

So

$$x(t) = \frac{1}{4\Omega^3}\left(e^{\Omega t}H(t) - e^{-\Omega t}H(t)\right) - \frac{1}{2\Omega^2}\left(\frac{1}{\Omega}H(t)\sin(\Omega t)\right)$$

$$= \frac{1}{4\Omega^3}H(t)\left(e^{\Omega t} - e^{-\Omega t} - 2\sin(\Omega t)\right)$$

$$= \frac{1}{2\Omega^3}H(t)\left(\sinh(\Omega t) - \sin(\Omega t)\right)$$